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On the fluctuations of the Casimir force: II. The stress-correlation function

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Abstract. The quantized Maxwell field in a halfspace exerts stresses on the boundary plane, whose zero-point fluctuations are analysed for a perfectly-conducting surface. The stress-correlation function on the surface is determined; its Fourier transform with respect to time and distance governs the mean-square deviation of the stress averaged over finite areas (diameters of order a) and over finite times of order T. Conditions guaranteeing physically sensible results are specified on the Fourier transforms of the averaging functions, with a systematic expansion for $a/cT \ll 1$, and the first few terms of an asymptotic approximation for $a/cT \gg 1$. The small-time behaviour of the surface-averaged correlation function appears to present a paradox, which is elucidated. Finally, a very simple expression is found to leading order for the mean-square fluctuating force on perfectly conducting large bodies of arbitrary shape.

1. Introduction

1.1. Preliminaries

The Casimir attractive force ($\pi^2/240\, L^4$ per unit area between infinitely extended, parallel, perfectly conducting mirrors a distance L apart) is governed by the vacuum expectation-values of the Maxwell stress tensor

$$S_{zz} \equiv (E_z^2 - B_x^2 - B_y^2)/8\pi \equiv S(t, \boldsymbol{r}) \tag{1.1}$$

on the mirror surfaces†. Neither $S(t, \boldsymbol{r})$ nor its integrals over all or part of any surface commute with the Hamiltonian of the quantized field; hence their measured values are in principle subject to fluctuations, whose analysis was addressed in paper I (Barton 1991). The physical motivation for the exercise was spelled out and referenced in I, and will not be repeated here; but we do repeat two important disclaimers. First, we deal only with mean-square deviations: in general, questions about the underlying probability distributions remain wide open. Second, the deviations we define and calculate, once found, turn out to be far too small to measure directly. However, the aim here as in I is to start to develop manageable routines for dealing with zero-point

† We use natural units $\hbar = 1 = c$, and unrationalized Gaussian units for the Maxwell field. Fields and stresses will be required only on the xy plane, whence we introduce the two-component vector $\boldsymbol{r} = (x, y)$, and its Fourier-conjugate $\boldsymbol{k} = (k_1, k_2)$.

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effects. The particular scenario we envisage is merely the simplest open to physically sensible analysis: though the ideas for dealing with it would certainly need to be extended before one could hope to make predictions verifiable in practice†, it seems likely that these ideas would also remain necessary.

Reverting then without further apology to the programme initiated in I, it remains the case that in I we used only the most immediate and elementary of mathematical methods, and our concern now is to put the calculation into more standard form, exploiting the appropriate stress-correlation function and its Fourier transform.

By making the results far more transparent and much easier to handle, this reformulation yields at least two advantages beyond the merely cosmetic (and beyond helping to correct numerical errors in some coefficients). First, one gains both physical and mathematical insight into convergence criteria, and into the asymptotics applicable when test bodies have characteristic linear dimensions a small or large compared with the duration T of a force measurement. Second, reasoning in terms of correlation functions makes it plain that to leading order our results for flat mirrors determine the fluctuating forces on large objects ($a \gg T$) of arbitrary shape, through simple geometric projection‡.

To elucidate the basic physics, we need consider only a single flat mirror, laterally infinite, and only one of its faces, say the xy plane, exposed to the zero-point fluctuations of the quantized Maxwell field in the half-space $z \geqslant 0$. To see why, note that by symmetry the mean force exerted on any part of the mirror across this surface is cancelled identically by the mean force exerted across the opposite surface (facing in the negative z-direction). By contrast, the fluctuations in the two halfspaces separated by the mirror are uncorrelated, whence, far from cancelling, the mean-square force deviations across opposite faces simply add. Thus it is indeed enough to do the calculation for only one, say only for the right-facing surface.

Our physical scenario is a measurement, extending over a finite time of order T, of the impulse imparted to a finite-area piston set flush with the mirror, with linear dimensions and area of orders a and $A \sim a^2$ respectively. The displacement of the piston during the measurement is ignored: roughly speaking, the piston is massive enough and T not too large. We shall see that this restricts the direct applicability of the results quite considerably, but it avoids over-complicating the calculations from the start.

1.2. Averaged stresses and the correlation function

To model mathematically the mean stress measured in this scenario, we introduce, first, a time average (identified by an overbar)

$$\bar{S}(\boldsymbol{r}) = \int_{-\infty}^{\infty} \mathrm{d}t\, f(t) S(t, \boldsymbol{r}). \tag{1.2}$$

The averaging function $f(t)$ mimics the finite duration of the measurement. Intuitively we think of it as a single peak of width $2T$; mathematically we take it to be real and

† One such problem that comes to mind is how to adapt the Einstein–Langevin theory of Brownian motion to neutral objects at absolute zero temperature.

‡ Small spheroidal objects are considered by Eberlein (1991). Such studies can yield sensible estimates of the fluctuations in practicable apparatus for measuring the Casimir effect.

subject to the following conditions:

$$f(t) \geqslant 0 \tag{1.3a}$$

$$\int_{-\infty}^{\infty} \mathrm{d}t\, f(t) = 1 \tag{1.3b}$$

$$\int_{-\infty}^{\infty} \mathrm{d}t\, f^2(t) = \mathrm{O}(1/T) < \infty. \tag{1.3c}$$

We shall also need the Fourier transform $g(\nu) = g^*(-\nu)$:

$$f(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi}\, g(\nu)\, \mathrm{e}^{-\mathrm{i}\nu t} \tag{1.4}$$

and recall Parseval's theorem

$$\int_{-\infty}^{\infty} \mathrm{d}\nu |g(\nu)|^2 = 2\pi \int_{-\infty}^{\infty} \mathrm{d}t\, f^2(t). \tag{1.5}$$

Similarly we introduce a position average (identified by a tilde), designed to mimic the finite dimensions of the piston:

$$\tilde{S}(t) = \int \mathrm{d}^2 r\, \phi(\boldsymbol{r}) S(t, \boldsymbol{r}) \tag{1.6}$$

where the real averaging function ϕ is thought of as a single peak, with linear dimensions a and area $A \sim a^2$. It satisfies the conditions

$$\phi(\boldsymbol{r}) \geqslant 0 \tag{1.7a}$$

$$\int \mathrm{d}^2 r\, \phi(\boldsymbol{r}) = 1 \tag{1.7b}$$

$$\int \mathrm{d}^2 r\, \phi^2(\boldsymbol{r}) = \mathrm{O}(1/A) < \infty \tag{1.7c}$$

$$\phi(\boldsymbol{r}) = \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\, \gamma(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) \tag{1.8a}$$

$$\int \mathrm{d}^2 k |\gamma(\boldsymbol{k})|^2 = (2\pi)^2 \int \mathrm{d}^2 r\, \phi^2(\boldsymbol{r}) = \mathrm{O}(1/A) < \infty. \tag{1.8b}$$

Note the special case where ϕ equals $1/A$ over a region of area A, and zero elsewhere; then the expressions in (1.8b) equal $(2\pi)^2/A$. Note also that on inversion (1.4) and (1.8a) entail $|g(\nu)| \leqslant g(0) = 1$ and $|\gamma(\boldsymbol{k})| \leqslant \gamma(0) = 1$.

Purely for illustration we shall sometimes choose the Lorentzian

$$f(t) = \frac{T/\pi}{t^2 + T^2} \qquad g(\nu) = \exp(-|\nu| T) \tag{1.9}$$

but no profit derives from any similar concretization of ϕ.

Finally, we define the joint time- and surface-average

$$\tilde{\bar{S}} = \int \mathrm{d}^2 r\, \phi(\boldsymbol{r}) \bar{S}(\boldsymbol{r}) \tag{1.10a}$$

$$= \iint \mathrm{d}^2 r\, \mathrm{d}t\, \phi(\boldsymbol{r}) f(t) S(t, \boldsymbol{r}). \tag{1.10b}$$

It is $\tilde{\bar{S}}$, with suitably chosen f and ϕ, that is, in principle, measurable, and whose fluctuations we ultimately require. It proves central to the problem that (without a cutoff) the mean-square deviations of S itself, and of $\tilde{S}$, are irrecoverably divergent. By contrast, we shall see that the mean-square deviations of the time-averaged $\bar{S}$ and $\bar{S}$ are well-defined provided $f(t)$ is smooth enough, i.e. provided $|g(\nu\to\infty)|$ falls fast enough.

Notice that we are concerned throughout, as the physics dictates, with averages of S, i.e. with averages of squared fields; emphatically, we are not concerned with squares of averaged fields.

The mean-square deviation of $\tilde{\bar{S}}$ in the vacuum state is

$$\Delta\tilde{\bar{S}}^2 \equiv \langle 0|\tilde{\bar{S}}^2|0\rangle - \langle 0|\tilde{\bar{S}}|0\rangle^2$$
$$= \tfrac{1}{2}\sum_{\lambda\lambda'} |\langle\lambda\lambda'|\tilde{\bar{S}}|0\rangle|^2 \tag{1.11}$$

where $|\lambda\lambda'\rangle$ denotes a two-photon state†, λ and λ' being shorthand labels for the Maxwell normal-modes in our half-space $z\geqslant 0$. Specifically‡, $\lambda=(s,l,\boldsymbol{k})$, where $s=1,2$ is a polarization index, l is the wavenumber normal to the mirror $(0\leqslant l)$ and $\boldsymbol{k}=(k_1,k_2)$ is the wavevector parallel to the mirror $(-\infty<k_{1,2}<\infty)$. These limits are always understood even if not written. Accordingly, $\Sigma_\lambda\ldots=\Sigma_s\int \mathrm{d}l\int \mathrm{d}^2k\ldots$. We also write $\omega=(l^2+k^2)^{1/2}$, and ϕ for the (plane) polar angle of $\boldsymbol{k}$; and similarly for ω', ϕ'.

The canonical approach to an object like $\Delta\bar{S}^2$ is through the underlying (i.e. un-averaged) correlation function

$$W(\tau,\boldsymbol{\rho})\equiv\langle 0|\tfrac{1}{2}\{S(t,\boldsymbol{r}),S(t',\boldsymbol{r}')\}_+|0\rangle-\langle 0|S|0\rangle^2 \tag{1.12a}$$

$$\tau\equiv t-t' \qquad \boldsymbol{\rho}\equiv\boldsymbol{r}-\boldsymbol{r}' \tag{1.12b}$$

where $\{,\}_+$ is the anticommutator, and where $\langle 0|S(t,\boldsymbol{r})|0\rangle$ is of course independent of t and $\boldsymbol{r}$.

The two-photon matrix-elements $\langle\lambda\lambda'|S(t,\boldsymbol{r})|0\rangle$ are given in I; here we only quote (the corrected version of) the raw expression for W:

$$W(\tau,\boldsymbol{\rho})=\frac{1}{32\pi^6}\,\mathrm{Re}\int \mathrm{d}l\,\mathrm{d}^2k\int \mathrm{d}l'\,\mathrm{d}^2k'\,\frac{1}{\omega\omega'}$$
$$\times\{l^2l'^2\cos^2(\phi-\phi')+(l^2\omega'^2+\omega^2l'^2)\sin^2(\phi-\phi')$$
$$+[-kk'+\omega\omega'\cos(\phi-\phi')]^2\}$$
$$\times\exp\{\mathrm{i}(\omega+\omega')\tau-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{\rho}\}. \tag{1.13}$$

We adopt the notation

$$\bar{W}(\rho)\equiv\iint \mathrm{d}t\,\mathrm{d}t'\,f(t)f(t')\,W(\tau,\boldsymbol{\rho}) \tag{1.14a}$$

$$\bar{W}(0)=\iint \mathrm{d}t\,\mathrm{d}t'\,f(t)f(t')\,W(\tau,0)=\Delta\bar{S}^2 \tag{1.14b}$$

† The factor $\frac{1}{2}$ enters because photons are indistinguishable. It was overlooked (albeit not quite consistently) in I, which led to wrong numerical coefficients there in equations (4.3), (4.5), (4.10*b*), (4.12), (4.13), (5.3), and (5.4*a*). The footnote on page 1000 suffered from an analytical error as well. Correct versions will emerge below.

‡ The normal-mode amplitudes are given in I.

where $\Delta\bar{S}^2$ is the ordinary vacuum-state mean-square deviation of the time-averaged stress $\bar{S}$, equation (1.2). In the special case $f(t)=\delta(t)$ (where $g(v)=1$) (1.14a) reduces to the equal-time correlation function $W(0,\rho)$. Similarly

$$\tilde{W}(\tau)\equiv\iint \mathrm{d}^2r\,\mathrm{d}^2r'\,\phi(\boldsymbol{r})\phi(\boldsymbol{r}')\,W(\tau,\boldsymbol{\rho}). \tag{1.15}$$

In the special case $\phi(\boldsymbol{r})=\delta(\boldsymbol{r})$ (where $\gamma(\boldsymbol{k})=1$) this reduces to the autocorrelation function at given point, i.e. to $W(\tau,0)$. (There is no surface-averaged analogue to (1.14b), because $\tilde{W}(\tau\to 0)$ diverges, as we shall see in section 4.2.)

Finally we define the overall average of W, which is precisely the mean-square deviation already defined in (1.11):

$$\tilde{\bar{W}}\equiv\iint \mathrm{d}t\,\mathrm{d}t'\iint \mathrm{d}^2r\,\mathrm{d}^2r'\,f(t)f(t')\,\phi(\boldsymbol{r})\phi(\boldsymbol{r}')\,W(\tau,\boldsymbol{\rho})=\Delta\tilde{\bar{S}}^2. \tag{1.16}$$

It is essential to remember that $\bar{W}(\rho)$ is a double average of $W(\tau,\boldsymbol{\rho})$ with respect to t and t', albeit with the same averaging function f, and similarly for $\tilde{W}$ and $\tilde{\bar{W}}$; emphatically, they are not averages directly, and only, with respect to the relative coordinates τ and/or $\boldsymbol{\rho}$.

1.3. Preview

The layout of the rest of this paper and of its main conclusions is as follows.

Section 2 evaluates the primary expression (1.13) for W, putting it into the succinct and elegant form $W=\mathrm{Re}\,6/\pi^4[(\tau+\mathrm{i}\varepsilon)^2-\rho^2]^4$. Its Fourier transform Γ (equations (2.6) and (2.7)) is found in appendix A, by three different arguments. The first two require somwhat abstruse integrals over products of Bessel functions; possibly they are illuminated by the third method, which though more circuitous mathematically is easier, and whose physics is much more transparent, in that it determines Γ directly from familiar Green functions for the Klein–Gordon or for the wave equation. The expression for Γ leads directly to our centrepiece expression (2.8) for $\tilde{\bar{W}}=\Delta\tilde{\bar{S}}^2$ in terms of the most general admissible averaging functions g and γ.

Section 3.1 uses $\tilde{\bar{W}}$ to establish some basic properties of $\Delta\tilde{\bar{S}}^2$. Section 3.1 discusses the conditions on g and γ under which $\Delta\tilde{\bar{S}}^2$ is well-defined. Section 3.2 considers the regime $a/T\ll 1$, derives the leading term $\Delta\tilde{\bar{S}}^2\sim 1/T^8$, equation (3.2), and also the systematic expansion (3.4). In section 3.2 we consider the opposite regime $a/T\gg 1$, where the leading term is $\Delta\tilde{\bar{S}}^2\sim 1/AT^6$, equation (3.5), and we explain how the proportionality to $1/A$ is in effect a consequence of the central limit theorem. In this regime one can go only two steps beyond the leading term (as in (3.6)) without disproportionately heavy analysis.

Section 4 reverts from $\tilde{\bar{W}}$ to the partially-averaged correlation functions. We give convenient integral representations for $\bar{W}(\rho)$ and $\tilde{W}(\tau)$ in terms, respectively, of the Fourier transforms $g(\nu)$ and $\gamma(\boldsymbol{k})$ of the averaging functions. It turns out that $\tilde{W}(\tau\to 0)$ is proportional to $-1/A\tau^6$, which seems paradoxical because $\tilde{W}(\tau)$ when further averaged over time must reproduce the positive-definite $\tilde{\bar{W}}=\Delta\tilde{\bar{S}}^2$. The last part of section 4.2 resolves the paradox, and spells out the physics of this minus sign.

For the regime $a\gg T$, section 5.1 generalizes the leading term of the mean-square force on a piston to large bodies of arbitrary shape, governed by (5.2). The underlying assumptions are elucidated in section 5.2 in terms of the stress-correlation function appropriate on the surface of such a body.

As already mentioned, appendix A derives the Fourier transform of W. Appendix B considers whether the uncertainty relation applied to the test piston allows $\Delta\bar{S}^2$ to be measured at least in principle, if as we have done one requires also that the displacement of the piston during the measurement be negligible. (Without this restriction the calculations, though basically no different, would become much more complicated.) The answer turns out to be no when $a \ll T$, and yes when $a \gg T$.

2. The stress-correlation function and its Fourier transform

In order to evaluate $W(\tau, \boldsymbol{\rho})$, equation (1.13), we set $l^2 = \omega^2 - k^2$, $l'^2 = \omega'^2 - k'^2$, and then employ the following trick. First we replace $\exp\{i(\omega+\omega')\tau - i(\boldsymbol{k}+\boldsymbol{k}')\rho\}$, temporarily†, by $\exp\{i(\omega\tau+\omega'\tau') - i(\boldsymbol{k}\cdot\boldsymbol{\rho}+\boldsymbol{k}'\cdot\boldsymbol{\rho}')\}$. Next, we replace ω by $-i\partial/\partial\tau$ and $\boldsymbol{k}$ by $i\boldsymbol{\nabla} \equiv i(\partial/\partial\rho_x, \partial/\partial\rho_y)$; similarly ω' and $\boldsymbol{k}'$ are replaced by $-i\partial/\partial\tau'$ and $i\boldsymbol{\nabla}'$ respectively. Finally of course we shall set $\tau = \tau'$ and $\boldsymbol{\rho} = \boldsymbol{\rho}'$.

Some manipulation yields

$$W(\tau, \boldsymbol{\rho}) = \frac{1}{32\pi^6} \operatorname{Re} Q\mathscr{L}(\tau, \boldsymbol{\rho})\mathscr{L}(\tau', \boldsymbol{\rho}')|_{\tau=\tau', \boldsymbol{\rho}=\boldsymbol{\rho}'} \tag{2.1}$$

where

$$Q \equiv \left(\frac{\partial^2}{\partial\tau^2} - \nabla^2\right)\left(\frac{\partial^2}{\partial\tau'^2} - \nabla'^2\right) + \left(\frac{\partial^2}{\partial\tau\partial\tau'} - \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}'\right)^2 \tag{2.2}$$

and

$$\mathscr{L}(\tau, \boldsymbol{\rho}) \equiv \int_0^\infty dl \int d^2k \frac{1}{\omega} \exp(i\omega\tau - i\boldsymbol{k}\cdot\boldsymbol{\rho}) \exp(-\varepsilon\omega) \tag{2.3}$$

$$\mathscr{L}(\tau, \boldsymbol{\rho}) = -2\pi \frac{1}{(\tau + i\varepsilon)^2 - \rho^2}. \tag{2.4}$$

The convergence factor $\exp(-\varepsilon\omega)$ in (2.3) is supplied for convenience; the limit $\varepsilon \to 0^+$ is to be taken at the end of the calculation, when it yields mathematically sensible answers to physically sensible questions.

The differentiations prescribed by Q are simple though tedious. The end-result reads

$$\begin{aligned} W &= \frac{6}{\pi^4} \operatorname{Re} \frac{1}{[(\tau+i\varepsilon)^2 - \rho^2]^4} \\ &= \frac{6}{\pi^4}\frac{1}{2}\left\{\frac{1}{[(\tau+i\varepsilon)^2-\rho^2]^4} + \frac{1}{[(\tau-i\varepsilon)^2-\rho^2]^4}\right\}. \end{aligned} \tag{2.5}$$

The Fourier transform Γ of W is defined by

$$W(\tau, \boldsymbol{\rho}) = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \exp(i\Omega\tau) \int \frac{d^2K}{(2\pi)^2} \exp(-i\boldsymbol{K}\cdot\boldsymbol{\rho})\Gamma(\Omega, \boldsymbol{K}). \tag{2.6}$$

† Note that the distinction between (τ, ρ) and (τ', ρ') has nothing to do with the difference between the arguments $(t, \boldsymbol{r})$ and $(t', \boldsymbol{r}')$ in (1.12).

Appendix A inverts this (in the sense of generalized functions (Lighthill 1958)) to yield†

$$\Gamma(\Omega, \boldsymbol{K}) = \frac{1}{60\pi^2}\,\theta(\Omega^2 - K^2)(\Omega^2 - K^2)^{5/2} \tag{2.7}$$

where θ is the Heaviside step-function.

We emphasize that the apparent symmetry in (2.5) as between τ and ρ is wholly deceptive; (2.7) is ample evidence that the dependence on τ and the dependence on ρ are quite different, a difference that will prove basic to our analysis.

Finally, to evaluate the overall average $\bar{\bar{W}}$ we simply express W in (1.16) by (2.6), (2.7), and the f's and ϕ's likewise as Fourier integrals, by (1.4) and (1.8). Then we change to barycentric and relative variables, i.e. to $(t+t')/2 \equiv \chi$ and $(t-t') \equiv \tau$, and similarly to $(\boldsymbol{r}+\boldsymbol{r}')/2 \equiv \boldsymbol{\sigma}$ and $(\boldsymbol{r}-\boldsymbol{r}') \equiv \boldsymbol{\rho}$. Integration over the barycentric coordinates introduces factors $2\pi\delta(\nu+\nu')$ and $(2\pi)^2\delta(\boldsymbol{k}+\boldsymbol{k}')$, and leads to

$$\Delta\bar{\bar{S}}^2 = \bar{\bar{W}} = \frac{1}{240\pi^5}\int_0^\infty d\nu |g(\nu)|^2 \int d^2k |\gamma(\boldsymbol{k})|^2 \theta(\nu - k)(\nu^2 - k^2)^{5/2} \tag{2.8}$$

which is the central result of this paper.

3. The mean-square averaged stress

3.1. Convergence

The necessary and sufficient condition for $\bar{\bar{W}}$ to be well-defined, i.e. for (2.8) to converge, reads

$$\int^\infty d\nu\, |g(\nu)|^2 \nu^5 < \infty. \tag{3.1}$$

Recall that the behaviour of $g(\nu \to \infty)$ is governed by the singularities of $f(t)$ at finite t. Singularities on the real axis are discussed, for example, by Lighthill (1958). If f has no singularities for $|\mathrm{Im}\, t| < \eta$, then $g(\nu \to \infty) \leqslant \exp(-\eta\nu)$.

To understand (3.1), note‡ that for fixed ν in (2.8) the inner integral $\int d^2k$ converges trivially because $k < \nu$; hence only the convergence of $\int^\infty d\nu$ is in question, i.e. only the behaviour of the integrand at arbitrarily large ν. Since the convergence of $\int d^2k |\gamma|^2$ is already posited in (1.8b), we can, for large ν in (2.8), replace $(\nu^2 - k^2)^{5/2} \to \nu^5$. The resultant inner integral is just $\nu^5 \int d^2k |\gamma|^2$, which validates the criterion (3.1).

Notice especially that this condition on g, i.e. on the time-averaging, cannot be relaxed through any further conditions imposed on γ, i.e. on the surface-averaging. Conversely, no convergence conditions are required on γ beyond those already adopted in (1.7) and (1.8). This is perhaps the most telling illustration of our earlier remark that the apparent symmetry between τ and ρ in W, equation (2.5), is deceptive.

3.2. Short distance, long time

In almost any realistically conceivable apparatus one would have $a \ll T$, i.e. piston diameters a much smaller than the distance cT that light travels during the measurement. Then ϕ is a much narrower peak than f; conversely γ is much flatter than g,

† $\Gamma(\Omega, \boldsymbol{K})$ must not of course be confused with gamma functions.

‡ We are talking our way through Fubini's and Tonelli's theorems (see, e.g., Weir 1973).

and to leading order in a/T we can approximate in (2.8) by setting $|\gamma(\boldsymbol{k})| \simeq \gamma(0) = 1$, because $|g|^2$ becomes negligible before $|\gamma(\boldsymbol{k})|^2$ can change appreciably from $|\gamma(0)|^2$. Formally, this limit corresponds to $\phi(\boldsymbol{r}) = \delta(\boldsymbol{r})$, whence it reduces $\bar{W} = \Delta\bar{S}^2$ to $\bar{W}(0) = \Delta\bar{S}^2$, as one can see from (1.10) and (1.14). Since $|\gamma|^2 = 1$ reduces the inner integral in (2.8) to $2\pi\nu^7/7$, we find†

$$\lim_{a/T\to 0} \bar{\bar{W}} = \lim \Delta\bar{\bar{S}}^2 = \frac{1}{840\pi^4}\int_0^\infty \mathrm{d}\nu |g(\nu)|^2 \nu^7 = \mathrm{O}(1/T^8) \tag{3.2}$$

where the final estimate follows dimensionally. With the Lorentzian (1.9) for instance, i.e. with $|g|^2 = \exp(-2\nu T)$, one obtains $\Delta\bar{\bar{S}}^2 = 3/(2^7\cdot\pi^4\cdot T^8)$.

Notice that (3.2) is just what one would have got by tackling directly the time-averaged stress at a given point, i.e. without any surface-averaging in the first place. The reason, pointed out in I, is that averages over times of order T are dominated by normal modes with frequencies $\omega \lesssim 1/T$, i.e. with wavelengths $\lambda \gtrsim T$; but these are *effectively coherent over distances up to order T, so that surface averaging over shorter* distances make no difference.

Of course (3.2) is merely the first term of the systematic expansion appropriate when $a/T \ll 1$, found by writing

$$\int_0^{2\pi} \mathrm{d}\phi |\gamma(\boldsymbol{k})|^2 = 2\pi\left\{\mu_0 + \mu_1(k_a) + \mu_2\frac{1}{2!}(ka)^2 + \ldots\right\} \tag{3.3}$$

where ϕ is the (plane) polar angle of $\boldsymbol{k}$ and the μ_n are dimensionless coefficients, with $\mu_0 = 1$. On substituting into (2.8) and integrating over k we obtain

$$\bar{\bar{W}} = \frac{\Gamma(\frac{7}{2})}{240\pi^4}\sum_{n=0}^{\infty}\frac{\mu_n a^n}{n!}\frac{\Gamma((n+2)/2)}{\Gamma((n+9)/2)}\int_0^\infty \mathrm{d}\nu |g|^2 \nu^{n+7} \tag{3.4}$$

which is effectively an expansion in powers of a/T, provided $g(\nu\to\infty)$ falls faster than any inverse power. For instance, Lorentzian averaging yields

$$\bar{\bar{W}} = \frac{\Gamma(\frac{7}{2})}{240\pi^4}\frac{1}{(2T)^8}\sum_{n=0}^{\infty}\mu_n\left(\frac{a}{2T}\right)^n\frac{\Gamma((n+2)/2)}{\Gamma((n+9)/2)}\frac{(n+7)!}{n!}$$

whose appearance suggests that such series probably have finite radii of convergence.

3.3. Long distance, short time

In the other extreme $a \gg T$, γ is much narrower than g. Then in (2.8) we can to leading order approximate by setting $(\nu^2 - k^2)^{5/2} \to \nu^5$, because $|\gamma|^2$ is already negligible before k becomes comparable with ν. Thus the inner integral becomes $\nu^5 \int \mathrm{d}^2 k\, |\gamma|^2$, and appeal to Parseval's theorem (1.8b) yields‡

$$\begin{aligned}\lim_{a/T\to\infty} \bar{\bar{W}} &= \lim \Delta\bar{\bar{S}}^2 = \frac{1}{60\pi^3}\left(\int_0^\infty \mathrm{d}\nu |(\nu)|^2 \nu^5\right)\int \mathrm{d}^2 r\, \phi^2(\boldsymbol{r}) \\ &= \mathrm{O}(1/AT^6)\end{aligned} \tag{3.5}$$

where the estimate again follows dimensionally. With Lorentzian time-averaging for instance, (3.5) gives $\Delta\bar{\bar{S}}^2 \approx (1/2^5\pi^3 T^6)\int \mathrm{d}^2 r\, \phi^2$.

† If (3.2) diverges one must revert to (2.8). But it is hard to imagine any plausible time-averaging function for which $\int^\infty \mathrm{d}\nu |g|^2\nu^5$ converges while $\int^\infty \mathrm{d}\nu |g|^2 \nu^7$ does not.

‡ The coefficient here corrects the footnote on p 1000 of I.

Systematic approximation is harder to come by than for $a \ll T$, and we settle for just two terms beyond (3.5), obtainable from (2.8) through the binomial expansion of $(\nu^2 - k^2)^{5/2}$:

$$\tilde{\bar{W}} = \frac{1}{240\pi^5} \int_0^\infty \mathrm{d}\nu |g(\nu)|^2 \nu^5 \int \mathrm{d}^2 k\, |\gamma(\boldsymbol{k})|^2 \theta(\nu - k) \left\{ 1 - \frac{5}{2}\frac{k^2}{\nu^2} + \frac{15}{8}\frac{k^4}{\nu^4} - \ldots \right\}. \tag{3.6}$$

Provided the k-integrals continue to converge we can approximate (3.6) by ignoring the cutoff $\theta(\nu - k)$; then it is clear dimensionally that the terms k^2/ν^2 and k^4/ν^4 generate factors T^2/a^2 and T^4/a^4 respectively. This simplistic approach can go no further, because $\int_0 \mathrm{d}\nu\, |g|^2/\nu$ diverges at its lower limit; on the other hand, more precise asymptotics would become far too elaborate.

The proportionality to $1/A$ in (3.5) stems directly from the central limit theorem. One can see this if one views $\tilde{\bar{S}}$ as in (1.10a), i.e. if one starts with the time-average $\bar{S}(\boldsymbol{r})$ and then regards $\tilde{\bar{S}}$ as the surface-average of $\bar{S}(\boldsymbol{r})$ over a large piston of area of order $A \sim a^2$. Roughly speaking, when $a \gg T$ the argument in the paragraph below (3.2) allows us to think of the piston as subdivided into very many patches, each with diameter of the order of the coherence length T of $\bar{S}$. The contributions to $\tilde{\bar{S}}$ from different patches have effectively random relative phases, and the number of patches is of order A/T^2. Then by the central limit theorem $\Delta\tilde{\bar{S}}^2$ is inversely proportional to this number, i.e. is proportional to $1/A$.

To justify this reasoning in more detail, we refer forward to section 4.1, which supplies the time-averaged correlation function, and to section 5.2, which explicates the underlying integrals. In anticipation, we record the mean-square force $\Delta\bar{F}^2$ stemming from (3.5) in the special case where $\phi(\boldsymbol{r})$ equals $1/A$ over a region of area A, and vanishes elsewhere. Then $\Delta\bar{F}^2 = (A\Delta\tilde{\bar{S}})^2$ simply from the definition of mean stress, while $\int \mathrm{d}^2 r\, \phi^2 = 1/A$. Consequently

$$\Delta\bar{F}^2 = (\alpha^2/T^6)A \tag{3.7a}$$

where α^2 is dimensionless, and α^2/T^6 merely a convenient shorthand for the proportionality constant, defined by

$$\alpha^2/T^6 \equiv \frac{1}{60\pi^3} \int_0^\infty \mathrm{d}\nu\, |g(\nu)|^2 \nu^5. \tag{3.7b}$$

Finally, to the extent that the approximations (3.5) and (3.7) are adequate, the fact that they derive from the central-limit theorem automatically establishes that the underlying probability distributions are Gaussians. Unfortunately, no such simplicity obtains in other regimes.

4. The partially-averaged correlation functions

4.1. The time-averaged correlation function $\bar{W}(\rho)$

This function, defined in (1.14), generalizes the equal-time correlation function $W(0, \rho) = 6/\pi^4\rho^8$, to which it reduces in the special case $f(t) = \delta(t)$.

One can get a fair preliminary idea about $\bar{W}(\rho)$ by evaluating it in the special case of the Lorentzian (1.9). Contour integrations with respect to t' and to t yield

$$\bar{W}(\rho) = \frac{6}{\pi^4} \frac{1}{(4T^2 + \rho^2)^4} \qquad \text{(Lorentzian)}. \tag{4.1}$$

In the general case we tackle (1.4) by substituting for $f(t)$ and $f(t')$ from (1.4), and by setting

$$W(\tau, \boldsymbol{\rho}) = (1/\pi^4)\,\mathrm{Re}(\partial/\partial\rho^2)^3[(\tau+\mathrm{i}\varepsilon)^2-\rho^2]^{-1}.$$

Again we change to barycentric and relative variables, and find

$$\bar{W}(\rho) = \iint \frac{\mathrm{d}\nu\,\mathrm{d}\nu'}{(2\pi)^2} g(\nu)g(\nu') \iint \mathrm{d}\chi\,\mathrm{d}\tau \times \exp\{-\mathrm{i}(\nu+\nu')\chi - \mathrm{i}\tau(\nu-\nu')/2\} \frac{1}{\pi^4}\left(\frac{\partial}{\partial\rho^2}\right)^3 \mathrm{Re}\frac{1}{(\tau+\mathrm{i}\varepsilon)^2-\rho^2} \tag{4.2}$$

where the correct placing of the instruction Re is important: it must not be taken outside the integral. Next, $\int \mathrm{d}\chi$ yields a factor $2\pi\delta(\nu+\nu')$, and we explicate the requisite real part as in (2.5). We can also set $\int_{-\infty}^{\infty}\mathrm{d}\nu = 2\int_0^\infty \mathrm{d}\nu$; this leads to

$$\bar{W}(\rho) = \frac{1}{\pi^5}\int_0^\infty \mathrm{d}\nu |g(\nu)|^2 \left(\frac{\partial}{\partial\rho^2}\right)^3 \int_{-\infty}^{\infty}\mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\nu\tau}\frac{1}{2}\left[\frac{1}{(\tau+\mathrm{i}\varepsilon)^2-\rho^2}+\frac{1}{(\tau-\mathrm{i}\varepsilon)^2-\rho^2}\right] \tag{4.3a}$$

$$= -\frac{1}{\pi^4}\int_0^\infty \mathrm{d}\nu |g(\nu)|^2 \left(\frac{\partial}{\partial\rho^2}\right)^3 \frac{\sin(\rho\nu)}{\rho}. \tag{4.3b}$$

Recognizing the spherical Bessel function $\sin(z)/z = j_0(z)$, and recalling $(\partial/\partial z^2)^3 j_0(z) = -j_3(z)/(2z)^3$, we obtain

$$\bar{W}(\rho) = \frac{1}{8\pi^4}\int_0^\infty \mathrm{d}\nu |g(\nu)|^2 \nu^4 \frac{j_3(\nu\rho)}{\rho^3} \tag{4.3c}$$

$$= \frac{1}{\rho^8}\frac{1}{8\pi^4}\int_0^\infty \mathrm{d}x |g(x/\rho)|^2 x^4 j_3(x) \tag{4.3d}$$

$$= \frac{1}{\rho^8}\frac{1}{8\pi^4}\left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty \mathrm{d}x |g(x/\rho)|^2 x^{7/2} J_{7/2}(x). \tag{4.3e}$$

It is easy to check that (4.3b) reduces as it should to (4.1) when $|g|^2 = \exp(-2\nu T)$, and that (4.3c) reduces to (3.2) as $\rho \to 0$, in the sense that $|g(\nu)|^2$ is already small before $\nu\rho$ becomes comparable with 1.

In the regime $\rho \to \infty$ (matching the approximation $f(t) \approx \delta(t)$) it is clear directly from the definition (1.14a) that

$$\bar{W}(\rho\to\infty) \approx 6/\pi^4\rho^8. \tag{4.4}$$

By contrast, it is not so obvious how to recover this conclusion directly† from (4.3c–e). We argue that in this limit $|g(x/\rho)|^2$ in (4.3e) functions merely as a convergence factor, allowing one to evaluate the integral by continuation in the indices m, n from the convergent case,

$$\int_0^\infty \mathrm{d}x\, x^m J_n(x) = 2^m \frac{\Gamma((n+m+1)/2)}{\Gamma((n-m+1)/2)} \tag{4.5}$$

(Abramowitz and Stegun (1965), (11.4.16)). On setting $m = n = \frac{7}{2}$ we do indeed recover (4.4).

† For instance, one gets nowhere with the attempt to use the large-argument asymptotic form of $j_3(\nu\rho)$ in (4.3c).

4.2. *The surface-averaged correlation function* $\tilde{W}(\tau)$

One standard way of exploring Casimir fluctuations, at least in principle, would be to study the autocorrelation function (in time) of the force experienced by a finite piston; or in other words to study the surface-averaged correlation function defined in (1.14).

We start with the more realistic case where $a \ll \tau$. Then (2.5) makes it clear, directly in configuration space, that

$$\tilde{W}(\tau \to \infty) \approx 6/\pi^4 \tau^8 \tag{4.6}$$

an evident counterpart of (4.4).

The other extreme $\tau \ll a$ produces some surprises; it is interesting if only because it illustrates some subtleties concerning $W(\tau, \boldsymbol{\rho})$ that are not immediately apparent from (2.5) on mere inspection.

We start by deriving for $\tilde{W}(\tau)$ an integral representation analogous to (4.3) for $\bar{W}(\rho)$. Substituting into (1.15) the Fourier representations (1.8a) of the ϕ's and (2.6), (2.7) of W, we are led to

$$\tilde{W}(\tau) = \frac{1}{(2\pi)^3 60\pi^2} \int \mathrm{d}^2K |\gamma(\boldsymbol{K})|^2 2\,\mathrm{Re} \int_K^\infty \mathrm{d}\Omega (\Omega^2 - K^2)^{5/2} \exp[-\mathrm{i}\Omega(\tau - \mathrm{i}\varepsilon)] \tag{4.7a}$$

$$\tilde{W}(\tau) = \frac{1}{32\pi^4 \tau^3} \int \mathrm{d}^2K |\gamma(\boldsymbol{K})|^2 K^3 Y_3(\tau K) = \frac{1}{32\pi^4 \tau^8} \int \mathrm{d}^2\kappa |\gamma(\boldsymbol{\kappa}/\tau)|^2 \kappa^3 Y_3(\kappa) \tag{4.7b}$$

where Y_3 is a Bessel function of the second kind. So far, (4.7) is exact.

The limit $\tau \to \infty$, equation (4.6), emerges from (4.7b) if one argues as we did about $\bar{W}(\rho)$ at the end of section 4.1 above. Mathematically more straightforward but physically more startling is the other extreme $\tau/a \to 0$, where $|\gamma(\boldsymbol{\kappa}/\tau)|^2$ falls very fast with increasing κ. Hence we replace $\kappa^3 Y_3(\kappa)$ by $\lim_{\kappa \to 0} \kappa^3 Y_3(\kappa) = -16/\pi$, noting the negative sign. One finds

$$\begin{aligned}
\tilde{W}(\tau \to 0) &\approx -\frac{1}{2\pi^5 \tau^8} \int \mathrm{d}^2\kappa |\gamma(\boldsymbol{\kappa}/\tau)|^2 \\
&= -\frac{1}{2\pi^5 \tau^6} \int \mathrm{d}^2K |\gamma(\boldsymbol{\kappa})|^2 \\
&= -\mathrm{Re} \frac{2}{\pi^3 (\tau + \mathrm{i}\varepsilon)^6} \int \mathrm{d}^2 r\, \phi^2.
\end{aligned} \tag{4.8}$$

In the last step have used Parseval's theorem (1.8b). We have also reverted to the explicit form $(\tau + \mathrm{i}\varepsilon)$.

The negative sign in (4.8) might seem paradoxical, because we know that the average of $\tilde{W}(\tau)$ over $f(t)$ and $f(t')$ (subject here to $a \ll T$) must be positive, being just the mean-square deviation $\Delta \tilde{S}^2$. Specifically, this average must yield (3.5). In fact the formalism warns one to be careful, since the limits $\tau \to 0$ and $\varepsilon \to 0$ are manifestly incompatible. For fixed finite τ however small, $\varepsilon \to 0$ in (4.8) yields negative $\tilde{W}(\tau)$, indicating strong anticorrelation between values of the stress at closely neighbouring times, even after surface averaging. (This is just another reminder of the violence of zero point fluctuations generally, which it is the object of the present paper, as it was of I, to render amenable to rational analysis.) By contrast, if one tried to consider directly the mean-square local stress at a given time, then one would need to take the limit $\tau \to 0$ first and $\varepsilon \to 0$ only afterwards, i.e. strictly 'at the end of the calculation', according to the letter of the law. This limit, however, is both mathematically and physically ill-defined: at a given point it diverges like $1/\varepsilon^8$, and even surface-averaging softens it only from $1/\varepsilon^8$ to $1/\varepsilon^6$.

To resolve the apparent paradox, i.e. to show that the formalism is self-consistent, one must verify that the appropriate average of $\mathrm{Re}[1/(\tau+\mathrm{i}\varepsilon)^6]$ is negative. To this end we write

$$\overline{1/\tau^6} \equiv \mathrm{Re}\,\overline{1/(\tau+\mathrm{i}\varepsilon)^6} \equiv \iint \mathrm{d}t\,\mathrm{d}t'\,f(t)f(t')\,\mathrm{Re}\frac{1}{(t-t'+\mathrm{i}\varepsilon)^6} \tag{4.9a}$$

$$= \iint \frac{\mathrm{d}\nu\,\mathrm{d}\nu'}{(2\pi)^2} g(\nu)g(\nu') \iint \mathrm{d}\chi\,\mathrm{d}\tau$$

$$\times \exp\{-\mathrm{i}(\nu+\nu')\chi - \mathrm{i}\tau(\nu-\nu')/2\}\,\mathrm{Re}\frac{1}{(\tau+\mathrm{i}\varepsilon)^6}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\nu |g(\nu)|^2 \int_{-\infty}^{\infty} \mathrm{d}\tau\,\mathrm{e}^{-\mathrm{i}\nu\tau}\frac{1}{2}\left[\frac{1}{(\tau+\mathrm{i}\varepsilon)^6}+\frac{1}{(\tau-\mathrm{i}\varepsilon)^6}\right]. \tag{4.9b}$$

Again, the correct placing of the instruction Re is important. Contour integration yields

$$\overline{1/\tau^6} = \frac{1}{\pi}\int_0^{\infty} \mathrm{d}\nu |g(\nu)|^2 \tfrac{1}{2}(-2\pi\mathrm{i})\frac{(-\mathrm{i}\nu)^5}{5!}$$

$$= -\frac{1}{5!}\int_0^{\infty} \mathrm{d}\nu |g(\nu)|^2 \nu^5 \tag{4.10}$$

with the requisite minus sign. Substitution into (4.8) then reproduces (3.5) as it must.

Loosely speaking one might perhaps describe what has happened by saying that the negative contribution to the average $1/\tau^6$ from the singularity itself (i.e. from the region $\tau<\varepsilon$) outweighs the positive contributions from $\tau>\varepsilon$.

5. Mean-square forces on large bodies or arbitrary shape

5.1. Plausibility argument and end-result

Consider the finite-time-averaged force $\bar{F}_u \equiv \bar{\boldsymbol{F}}\cdot\boldsymbol{u}$ in the direction of the unit vector $\boldsymbol{u}$, exerted on part or all of the surface of a perfectly-conducting body, by the zero-point fluctuations in the surrounding vacuum. For an isolated body as a whole $\langle 0|\bar{F}_u|0\rangle$ vanishes by translation invariance; but the mean-square deviation $\Delta\bar{F}_u^2 \equiv \langle 0|\bar{F}_0^2|0\rangle - \langle 0|\bar{F}_u|0\rangle^2$ does not vanish, and its value depends on $\boldsymbol{u}$ except in the very special case of the force on the entire surface of a sphere.

We are interested in $\Delta\bar{F}_u^2$ for a (generally curved) surface region A; we shall use the same symbol A also for the actual area of the region, and the symbol a generically for any linear dimensions characterizing A. Thus $A\sim a^2$, and we restrict ourselves to surfaces with principal radii of curvature of order not less than a. Simple examples are (all or parts of) ellipsoids of moderate eccentricity. We can include objects like a solid half-sphere, on the mathematical assumption that the leading term of $\Delta\bar{F}_u^2$ is wholly accounted for by the appropriate surface integrals, with no special contribution to this order from sharp rims† whose opening angles are finite. As before, large and small a will mean $a \gg T$ and $a \ll T$ respectively.

Since the tangential stresses vanish, we retain $S(\boldsymbol{r})$ to denote the time-averaged normal stress at a point $\boldsymbol{r}$ on the surface. Writing a typical vector surface-element as $\mathrm{d}\boldsymbol{A} = \mathrm{d}A\,\boldsymbol{n}$, with $\boldsymbol{n}$ the unit outward normal, we have

$$\bar{F}_u = -\int_A \mathrm{d}A\,\bar{S}(\boldsymbol{r})(\boldsymbol{n}\cdot\boldsymbol{u}). \tag{5.1}$$

When $a \lesssim T$, the calculation of $\Delta\bar{F}_u^2$ even with simple shapes is quite demanding; for small spheroids and for spheres of any size it is considered by Eberlein (1991). But for large bodies, $\Delta\bar{F}_u^2$ follows almost at once if we assume that, locally and to leading order, the zero-point fluctuations on such surfaces are the same as they would be on an infinite conducting plane coincident with the local tangent plane. Then the discussion in section 3.3 makes it plausible that a large surface region A can be subdivided into very many elements $\mathrm{d}A \ll A$, which are however still large enough individually to contribute to $\Delta\bar{F}_u^2$ proportionately to $\mathrm{d}A$; the proportionality constants are essentially the same as for a truly flat surface, i.e. the same as in (3.7), but weighted now with the purely geometric factor $(\boldsymbol{n}\cdot\boldsymbol{u})^2$. This factor allows for the gradually varying inclination of the local tangent plane to the direction $\boldsymbol{u}$ of the requisite force component.

Section 5.2 will describe just what these assumptions mean in terms of the correlation function. If they are accepted, then they yield the end-result immediately, and in the very simple form

$$\Delta\bar{F}_u^2 = (\alpha^2/T^6)\int_A \mathrm{d}A(\boldsymbol{n}\cdot\boldsymbol{u})^2. \tag{5.2}$$

For a sphere of radius $R \gg T$, and writing Ω for solid angle, we get

$$\Delta\bar{F}_u^2 = \frac{\alpha^2}{T^6}\int \mathrm{d}\Omega\; R^2\cos^2\theta = \frac{4\pi\alpha^2}{3}\frac{R^2}{T^6}. \tag{5.3}$$

For the curved surface of a hemisphere, with $\boldsymbol{u}$ along the polar axis, it is clear from symmetry that (5.2) yields just half (5.3), ie. just $(2\pi\alpha^2/3)(R^2/T^6)$. These results may be compared with the contribution $(\pi\alpha^2)(R^2/T^6)$ from the flat surface of the hemisphere, and with the consequent total for the half-sphere, which reads $\alpha^2(\pi+2\pi/3)\times(R^2/T^6) = (5\pi\alpha^2/3)(R^2/T^6)$. It is amusing to observe that in fact $\Delta\bar{F}_u^2$ for the curved surface is independent of the angle between $\boldsymbol{u}$ and the polar axis, i.e. that it equals $(2\pi\alpha^2/3)(R^2/T^6)$ for all $\boldsymbol{u}$.

5.2. Derivation from the correlation function

Just what our assumptions really posit is perhaps best appreciated by explicating them in terms of the appropriate correlation function. This is defined by writing the expression on the right-hand side of (1.12*a*) as $W(\tau;\boldsymbol{r},\boldsymbol{r}')$, since it now depends on $\boldsymbol{r}$ and $\boldsymbol{r}'$ separately, and not only on their difference. Accordingly, its time average is written

† The problem of determining such special contributions reminds one of the classic asymptotic analyses of the Casimir stresses (i.e. of the $\langle 0|\bar{F}_u|0\rangle$ rather than of the $\Delta\bar{F}_u^2$): see, e.g., Baltes and Hilf (1976), and Balian and Duplantier (1978).

$\bar{W}(\boldsymbol{r}, \boldsymbol{r}')$. (We do not go into details about how precisely to coordinatize the surface.) We choose barycentric and relative coordinates as in section 2, and have $\bar{W} = \bar{W}(\boldsymbol{\sigma}+\boldsymbol{\rho}/2, \boldsymbol{\sigma}-\boldsymbol{\rho}/2)$. Define a function $H(\boldsymbol{r})$, equal to 1 on the region A, and 0 elsewhere. Then, in an obvious notation,

$$\Delta\bar{F}_u^2 = \int_A \int_A \mathrm{d}^2 r\, \mathrm{d}^2 r'\, \bar{W}(\boldsymbol{r}, \boldsymbol{r}')(\hat{\boldsymbol{r}}\cdot\boldsymbol{u})(\hat{\boldsymbol{r}}'\cdot\boldsymbol{u}) \tag{5.4a}$$

$$= \iint \mathrm{d}^2\sigma\, \mathrm{d}^2\rho\, H(\boldsymbol{\sigma}+\boldsymbol{\rho}/2)H(\boldsymbol{\sigma}-\boldsymbol{\rho}/2)$$

$$\times\, \bar{W}(\boldsymbol{\sigma}+\boldsymbol{\rho}/2, \boldsymbol{\sigma}-\boldsymbol{\rho}/2)(\widehat{\boldsymbol{\sigma}+\boldsymbol{\rho}/2}\cdot\boldsymbol{u})(\widehat{\boldsymbol{\sigma}-\boldsymbol{\rho}/2}\cdot\boldsymbol{u}) \tag{5.4b}$$

where hats identify unit vectors.

We are now in a position to spell out, in turn, the three assumptions (or approximations) we need, and their consequences. (a) Assume on geometrical grounds that $\bar{W}$ reduces to the form appropriate on the infinite flat boundary surface of a halfspace, as given in section 4.1. Then

$$\bar{W}(\boldsymbol{\sigma}+\boldsymbol{\rho}/2, \boldsymbol{\sigma}-\boldsymbol{\rho}/2) \approx \bar{W}(\rho)$$

where $\bar{W}(\rho)$ is given by (4.3), and for Lorentzians by (4.1). This is plausible if the radii of curvature much exceed the correlation length T of $\bar{W}$. (b) Approximate the product of the two H-functions by $H^2(\boldsymbol{\sigma}) = H(\boldsymbol{\sigma})$, and extend the integration over all values of $\boldsymbol{\rho}$. This is justified if the region is much wider than T, equation (3.5) validating the consequences explicitly. Thus

$$\Delta\bar{F}_u^2 \approx \int_A \mathrm{d}^2\sigma \int \mathrm{d}^2\rho\, \bar{W}(\rho)(\widehat{\boldsymbol{\sigma}+\boldsymbol{\rho}/2}\cdot\boldsymbol{u})(\widehat{\boldsymbol{\sigma}-\boldsymbol{\rho}/2}\cdot\boldsymbol{u}). \tag{5.5}$$

(c) Replace both scalar products by $\boldsymbol{n}(\boldsymbol{\sigma})\cdot\boldsymbol{u}$, with $\boldsymbol{n}(\boldsymbol{\sigma})$ the unit normal at $\boldsymbol{\sigma}$. This is justified (subject to (b)) if the radii of curvature are large and vary little over distances of order T, i.e. within the range of $\bar{W}$.

Under (a), (b), (c) jointly, (5.4) reduces to

$$\Delta\bar{F}_u^2 \approx \left(\int_A \mathrm{d}^2\sigma(\boldsymbol{n}\cdot\boldsymbol{u})^2\right)\left(\int \mathrm{d}^2\rho\, \bar{W}(\rho)\right) \tag{5.6}$$

where the second factor is a constant identified as α^2/T^6 by comparison with (3.7) for a flat surface. It is thus that we finally recover (5.2).

It may be worth stressing that higher-order corrections are now incomparably harder to identify than in section 3.3, because they now depend on the radii of curvature as well as on the size of the region A.

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Appendix A. The Fourier transform of W

We aim to establish (2.7) by inverting (2.6):

$$\Gamma(\Omega, K) \equiv \int d^2\rho \; e^{iK\cdot\rho} \int_{-\infty}^{\infty} d\tau \; e^{-i\Omega\tau} \, \mathrm{Re} \frac{6/\pi^4}{[(\tau+i\varepsilon)^2-\rho^2]^4} \tag{A.1a}$$

$$= \theta(\Omega^2-K^2) \frac{1}{60\pi^2} (\Omega^2-K^2)^{5/2}. \tag{A.1b}$$

Because the result is so crucial and the asymmetry between Ω and K perhaps surprising, we give three different derivations, two based on somewhat recondite lore about Bessel functions, and the third on familiar properties of relativistic Green's functions.

A.1. Bessel functions and brute force

Starting as in section 4.1, we write

$$\begin{aligned}\Gamma &= \frac{1}{\pi^4} \int d^2\rho \; e^{iK\cdot\rho} \left(\frac{\partial}{\partial\rho^2}\right)^3 \int d\tau \; e^{-i\Omega\tau} \frac{1}{2} \left\{ \frac{1}{(\tau+i\varepsilon)^2-\rho^2} + \frac{1}{(\tau-i\varepsilon)^2-\rho^2} \right\} \\ &= -\frac{1}{\pi^3} \int d^2\rho \; e^{iK\cdot\rho} \left(\frac{\partial}{\partial\rho^2}\right)^3 \frac{\sin\Omega\rho}{\rho} \\ &= \frac{\Omega^4}{8\pi^3} \int d^2\rho \; e^{iK\cdot\rho} \frac{1}{\rho^3} j_3(\Omega\rho) \\ &= \frac{\Omega^4}{8\pi^3} \left(\frac{\pi}{2\Omega}\right)^{1/2} \int d^2\rho \; e^{iK\cdot\rho} \frac{1}{\rho^{7/2}} J_{7/2}(\Omega\rho).\end{aligned} \tag{A.2}$$

Integration over the (plane) polar angle of ρ introduces a second Bessel function:

$$\Gamma(\Omega, K) = \frac{\Omega^4}{4\pi^2} \left(\frac{\pi}{2\Omega}\right)^{1/2} \int_0^{\infty} d\rho \frac{1}{\rho^{5/2}} J_0(K\rho) J_{7/2}(\Omega\rho). \tag{A.3}$$

The integral $\int_0^\infty d\rho \, \rho^l J_m(K\rho) J_n(\Omega\rho)$, discontinuous at $\Omega = K$, is discussed at length by Watson (1944). Equation (A.2) features a very special case, where $l = m - n + 1$. Then the integral vanishes if $\Omega^2 < K^2$, and one has (Abramowitz and Stegun (1965), equations (11.4.33), (11.4.34), (11.4.4))

$$\int_0^{\infty} d\rho \frac{1}{\rho^{5/2}} J_0(K\rho) J_{7/2}(\Omega\rho) = \theta(\Omega^2-K^2) \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{15\Omega^{7/2}} (\Omega^2-K^2)^{5/2}. \tag{A.4}$$

Substitution into (A.3) yields (A.1b) as promised.

A.2. Bessel functions and Parseval's theorem

The step function in (A.4) is surprising; its presence is explained, and the integral evaluated, by a beautifully direct argument given by Titchmarsh (1948). One need merely establish (for positive K and Ω) the two (symmetrically normed) Fourier sine transforms

$$J_0(K\rho) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} dy \sin(y\rho) \left\{ \frac{2^{1/2}}{\Gamma(1/2)} \frac{\theta(y-K)}{(y^2-K^2)^{1/2}} \right\} \tag{A.5}$$

$$\rho^{-5/2} J_{7/2}(\Omega\rho) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} dy \sin(y\rho) \left\{ \frac{1}{2^2\Gamma(3)} \frac{\theta(\Omega-y) y (\Omega^2-y^2)^2}{\Omega^{7/2}} \right\} \tag{A.6}$$

they are the inverses of relations found by exploiting the power-series definition of the Bessel functions.

Parseval's theorem then yields

$$\int_0^\infty \mathrm{d}\rho\{J_0(K\rho)\}\{\rho^{-5/2}J_{7/2}(\Omega\rho)\}$$

$$=\int_0^\infty \mathrm{d}y\,\theta(y-K)\theta(\Omega-y)\frac{2^{1/2}}{\Gamma(1/2)2^2\Gamma(3)}\frac{y(\Omega^2-y^2)^2}{\Omega^{7/2}(y^2-K^2)^2} \tag{A.7}$$

on evaluating this elementary integral we reproduce (A.4).

A.3. Relativistic method

In order to relate Γ to relativistic Green functions, we introduce and then integrate out† a third space-coordinate z, by acting on the right of (A.1a) with the identity

$$\int_{-\infty}^{\infty} \mathrm{d}K_3\,\delta(K_3)=\int_{-\infty}^{\infty}\frac{\mathrm{d}K_3}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}z\,\mathrm{e}^{\mathrm{i}K_3 z}.$$

We adopt the obvious four-vector notation

$$\begin{aligned}
&\vec{x}\equiv(x_0,\boldsymbol{x}) && \vec{k}\equiv(k_0,\boldsymbol{k})\equiv(\Omega,K_1,K_2,K_3)\\
&\vec{x}\cdot\vec{k}\equiv x_0k_0-\boldsymbol{x}\cdot\boldsymbol{k} && \vec{k}\cdot\vec{k}\equiv k_0^2-\boldsymbol{k}^2\\
&(x_0\pm\mathrm{i}\varepsilon)^2-\boldsymbol{x}^2=\vec{x}\cdot\vec{x}\pm\mathrm{i}\varepsilon x_0.
\end{aligned} \tag{A.8}$$

Then (A.1a) becomes

$$\Gamma(\Omega,K)=\int_{-\infty}^{\infty}\frac{\mathrm{d}K_3}{2\pi}\Lambda(\vec{k}) \tag{A.9}$$

$$\Lambda(\vec{k})=\frac{6}{\pi^4}\int \mathrm{d}^4x\,\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}}\frac{1}{2}\left\{\frac{1}{(\vec{x}\cdot\vec{x}+\mathrm{i}\varepsilon x_0)^4}+\frac{1}{(\vec{x}\cdot\vec{x}-\mathrm{i}\varepsilon x_0)^4}\right\} \tag{A.10}$$

where, by inspection, Λ is a Lorentz scalar, depending on $\vec{k}\cdot\vec{k}$ alone (and not also on the sign of k_0). We introduce one further auxiliary Lorentz-scalar variable λ^2, which allows us to write

$$\Lambda=\frac{1}{\pi^4}\lim_{\lambda\to0}\left(\frac{\partial}{\partial\lambda^2}\right)^3\int \mathrm{d}^4x\,\exp(-\mathrm{i}\vec{k}\cdot\vec{x})\frac{1}{2}\left\{\frac{1}{\vec{x}\cdot\vec{x}+\mathrm{i}\varepsilon x_0-\lambda^2}+\frac{1}{\vec{x}\cdot\vec{x}-\mathrm{i}\varepsilon x_0-\lambda^2}\right\}. \tag{A.11}$$

The essential step is to observe what happens if the roles of the variables $\vec{k}$ and $\vec{x}$ are interchanged, i.e. if one thinks of $\vec{k}$ as a position and of $\vec{x}$ as a momentum variable. Then the integral in (A.11) becomes instantly recognizable, and we see that

$$\Lambda=-16\lim_{\lambda\to0}\left(\frac{\partial}{\partial\lambda^2}\right)^3\frac{1}{2}\{D^{\mathrm{ret}}(\vec{k})+D^{\mathrm{adv}}(\vec{k})\}$$

† Basically this reverses Hadamard's 'method of descent', discussed in a related context, where it is called, more descriptively, 'the method of embedding' (Barton (1989), section 11.2.3).

featuring the familiar retarded and advanced Green's functions for the Klein–Gordon equation

$$\left(\frac{\partial^2}{\partial k_0^2}-\nabla_k^2+\lambda^2\right)D(\vec{k})=\delta(\vec{k})$$

in a Minkowski space with coordinates $\vec{k}$. Using the standard expression for the D's (Schwinger 1947; or see, for instance, Bogoliubov and Shirkov 1959) one finds

$$\Lambda=-16\lim_{\lambda\to 0}\left(\frac{\partial}{\partial\lambda^2}\right)^3\left\{\frac{1}{4\pi}\,\delta(\vec{k}\cdot\vec{k})-\theta(\vec{k}\cdot\vec{k})\,\frac{\lambda}{8\pi\sqrt{\vec{k}\cdot\vec{k}}}\,J_1(\lambda\sqrt{\vec{k}\cdot\vec{k}})\right\}. \qquad (\text{A.12})$$

Under $\lim_{\lambda\to 0}(\partial/\partial\lambda^2)^3$, the only term that survives is the third term of the expansion

$$J_1(z)=(z/2)-\frac{1}{1^2\cdot 2}(z/2)^3+\frac{1}{1^2\cdot 2^2\cdot 3}(z/2)^5-\ldots$$

which yields

$$\Lambda=\frac{1}{32\pi}\,\theta(\Omega^2-K_1^2-K_2^2-K_3^2)(\Omega^2-K_1^2-K_2^2-K_3^2)^2. \qquad (\text{A.13})$$

Substitution into (A.9) and integration over K_3 then reproduce (A.1b).

Instead of exploiting Schwinger's expression quoted in (A.12), one can derive (A.13) by evaluting (A.10) in conveniently chosen reference frames†, taking advantage of the fact that it depends only on $\vec{k}\cdot\vec{k}$. For spacelike $\vec{k}\cdot\vec{k}<0$, i.e. for $\boldsymbol{K}^2>\Omega^2$, one chooses the ('primed') frame where $k_0'=0$ and $\vec{k}\cdot\vec{k}=-\boldsymbol{k}'^2$. Then Λ becomes

$$\frac{1}{\pi^4}\int \mathrm{d}^3x\;\mathrm{e}^{i\boldsymbol{k}'\cdot\boldsymbol{x}}\left(\frac{\partial}{\partial \boldsymbol{x}^2}\right)^3\int_{-\infty}^{\infty}\mathrm{d}x_0\;\mathrm{Re}\,\frac{1}{(x_0+\mathrm{i}\varepsilon)^2-\boldsymbol{x}^2}=0$$

vanishing because $\int \mathrm{d}x_0$ vanishes. For timelike $\vec{k}\cdot\vec{k}>0$, i.e. for $\Omega^2>\boldsymbol{K}^2$, one chooses the ('double-primed') frame where $\boldsymbol{k}''=0$ and $\vec{k}\cdot\vec{k}=k_0''^2$. Integration over x_0 and manipulation along the lines of section 4.1 then lead to

$$\begin{aligned}\Lambda&=\frac{1}{\pi^4}\int \mathrm{d}^3x\left(\frac{\partial}{\partial \boldsymbol{x}^2}\right)^3\int \mathrm{d}x_0\;\mathrm{e}^{-\mathrm{i}k_0''x_0}\,\mathrm{Re}\,\frac{1}{(x_0+\mathrm{i}\varepsilon)^2-\boldsymbol{x}^2}\\&=\frac{1}{2\pi^2}\left(\frac{\pi}{2}\right)^{1/2}(k_0'')^4\int_0^{\infty}\mathrm{d}y\,y^{-3/2}J_{7/2}(y)\end{aligned} \qquad (\text{A.14})$$

which becomes applicable in arbitrary frames on restoring $k_0''^2=\vec{k}\cdot\vec{k}=\Omega^2-\boldsymbol{K}^2$.

The remaining integral (a pure number) is given by (4.5), and we recover (A.13).

Appendix B. Constraints from the uncertainty relation for the piston

Section 1.1 has already emphasized the severe practical limitations of the theoretical exercises in this paper and in I; here we consider whether any further limitations follow from the Heisenberg uncertainty relation for the piston or other test body (of mass m). It proves convenient to reserve, as hitherto, the symbol Δ for deviations due to

† This type of reasoning is spelled out elsewhere as a simple way to the propagators of the wave equation (Barton 1989, Appendix O).

the zero-point fluctuations of the Maxwell field; and to introduce the symbol δ for the familiar uncertainties of the piston position z and its conjugate momentum p, obeying the uncertainty relation $\delta z \cdot \delta p \geq \hbar/2$ as a consequence of the commutation rule $[z, p] = \mathrm{i}\hbar$, irrespective of any coupling between the piston and the field†.

We ask whether the δ-uncertainties allow the Δ-deviations to be observed in principle, *subject to the subsidiary condition* that the displacement of the piston during the measurement be negligible. We repeat from the end of section 1 that without this condition the calculations would proceed on basically the same lines but would need to become much more elaborate.

The mean-square impulse acquired by the piston from the field is

$$\Delta p^2 \sim a^4 \Delta \bar{\bar{S}}^2 T^2. \tag{B.1}$$

Obviously, Δp is directly measurable only if $\delta p \leq \Delta p$, and our problem is to ensure this (which automatically sets a lower bound on $\delta z \geq \hbar/\delta p$) while ensuring also that both the initial δz_i and the final δz_f are less than the coherence length cT, since otherwise it would have been wrong to ignore the displacement of the piston in calculating Δp^2. (For dimensional reasons the correlation length must be of order cT perpendicularly as well as parallel to the mirror.) It would be absurd of course to envisage position uncertainties as large in practice as any likely value of cT: the point is rather that this particular argument can afford to operate with very liberal bounds on the δz.

Accordingly, with $\delta p_i \geq \hbar/\delta z_i$, and taking account of distance travelled in time T at speed p/m, one estimates

$$\delta z_f \geq \delta z_i + T\delta p_i/m \geq \delta z_i + \hbar T/m\delta z_i. \tag{B.2}$$

However, since we can accept δz_i of order cT, equation (B.2) reduces to the very mild requirement $\delta z_f \geq (\partial z_i + \hbar/mc)$, where $\hbar/mc$ is the Compton wavelength for the piston, wholly negligible with respect to any conceivable value of cT. Thus we can afford to base further estimates on

$$\delta z_f \sim \delta z_i \sim \delta z \equiv cT \tag{B.3}$$

whence

$$\delta p_f \sim \delta p_i \sim \delta p \geq \hbar/\delta z = \hbar/cT. \tag{B.4}$$

The conclusions now follow immediately. In the regime $a/cT \ll 1$, equations (B.1) and (3.2) entail

$$\Delta p \sim \left[\frac{a^4\hbar^2T^2}{c^6T^8}\right]^{1/2} = \frac{a^2\hbar}{c^3T^3}$$

then $\Delta p > \delta p$ requires

$$a^2\hbar/c^3T^3 > \hbar/cT \Rightarrow (a/cT)^2 > 1 \tag{B.5}$$

which in this regime is false by definition.

By contrast, in the regime $a/cT \gg 1$, equations (B.1) and (3.5) ential

$$\Delta p \sim \left[\frac{a^4\hbar^2T^2}{c^4a^2T^6}\right]^{1/2} = \frac{a\hbar}{c^2T^2}$$

† In this appendix we restore $\hbar$ and c.

whence $\Delta_p > \delta p$ now reads

$$a\hbar/c^2T^2 > \hbar/cT \Rightarrow a/cT > 1 \tag{B.6}$$

which is perfectly consistent.

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